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Stable Bose-Einstein correlations

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Abstract The shape of Bose-Einstein (or HBT) correlation functions is determined for the case when particles are emitted from a stable source, obtained after convolutions of large number of elementary random processes. The two-particle correlation function is shown to have a stretched exponential shape, characterized by the Lévy index of stability $0 < \alpha \leq 2$ and the scale parameter R. The normal, Gaussian shape corresponds to a particular case, when $\alpha = 2$ is selected. The asymmetry parameter of the stable source, β is shown to be proportional to the angle, measured by the normalized three-particle cumulant correlations.

Key words particle correlations • Bose-Einstein correlations • HBT • Lévy distributions • multiparticle production in high energy particle and nuclear physics

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In high energy nuclear and particle physics, the space-time structure of particle emitting sources is often investigated with the help of the two-particle Bose-Einstein correlation functions. In heavy ion physics, these correlations are frequently called as HBT correlations to honor the astronomers R. Hanbury Brown and R. Q. Twiss, who invented a similar method [7] in radio astronomy to measure the angular diameter of main sequence stars.

The two-particle correlation function $C_2(\mathbf{k}_1,\mathbf{k}_2)$ is defined as

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{N_2(\mathbf{k}_1,\mathbf{k}_2)}{N_1(\mathbf{k}_1)N_1(\mathbf{k}_2)} \tag{1}$$

where $\mathbf{k}_i$ is the momentum of particle $i = 1,2$, and $N_1(\mathbf{k}_1)$ is the single particle invariant momentum distribution (IMD), while $N_2(\mathbf{k}_1,\mathbf{k}_2)$ is the two-particle invariant momentum distribution.

In this manuscript we highlight some of the results of Ref. [3], where we have investigated in great detail the Bose-Einstein or HBT correlation functions under the following three experimental conditions: i) the correlation function tends to a constant for large values of the relative momentum $q = k_1 - k_2$; ii) near $|q| = 0$, the correlation function deviates from its asymptotic, large $|q|$ value in a certain domain of its argument; iii) the two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Condition iii) is satisfied if the propagation of identical boson pairs from a chaotic (thermalized) source to the detector can be described by a plane wave approximation; this is possible if Coulomb and strong final state interactions as well as additional short range correlations e.g. caused by resonance decays can be corrected for or are negligible.

For clarity, let us consider first a one-dimensional, factorized toy model. Let x and k stand for the coordinate and momentum variables, respectively. The model is defined by the emission function

$$S(x,k) = f(x)\, g(k) \tag{2}$$

and the normalizations are

$$\int dx f(x) = 1, \quad \int dk g(k) = \langle n \rangle \tag{3}$$

where $<n>$ stands for the mean multiplicity. The single-particle spectrum is

$$N_1(k) = \int dx S(x,k) = g(k). \tag{4}$$

If iii) is valid, the Bose-Einstein symmetrized two-particle wave-function is

$$\psi_{k_1,k_2}(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\exp(ik_1x_1 + ik_2x_2) + \exp(ik_1x_2 + ik_2x_1)\right]. \tag{5}$$

In the Yano-Koonin formalism [12], the two-particle momentum distribution of chaotic sources is given as

$$N_2(k_1,k_2) = \int dx_1 dx_2 S(x_1,k_1)(x_2,k_2)\left|\psi_{k_1k_2}(x_1,x_2)\right|^2. \tag{6}$$

Let us introduce as auxiliary quantities the Fourier transformed source density distribution and the relative momentum as

$$\tilde{f}(q) = \int dx \exp(iqx) f(x), \qquad q = k_1 - k_2. \tag{7}$$

The two-particle Bose-Einstein correlation function is obtained as

$$C_2(k_1,k_2) = 1 + \left|\tilde{f}(q)\right| \tag{8}$$

that measures the absolute value squared Fourier transformed coordinate-space distribution function of the particle emitting source.

In physics, as well as in the theory of probability, the probability distribution of a sum of a large number of random variables is one of the important problems, and such distributions are frequently realized in Nature. Limit distributions characterize the probability distributions of random processes in the limiting case when the number of elementary independent random subprocesses tends to infinity. In case of high energy nuclear and particle physics, for example, the position of emission of an observable particle is obtained as a sum of a large number of position shifts due to various parton-parton scatterings, hadronization, rescattering of hadrons, and decay of hadronic resonances:

$$x = \sum_n x_n. \tag{9}$$

Hence the distribution of the sum is obtained as a n-fold convolution,

$$f(x) = \int dx_1 \ldots dx_n f_1(x_1) \ldots f_n(x_n) \delta(x - x_1 - x_2 \ldots - x_n). \tag{10}$$

Various forms of the Central Limit Theorem state, that under some conditions, the distribution of the sum of large number of random variables converges to a limit distribution. In case of normal elementary processes, the limit distribution of their sum is the Gaussian distribution. This is one of the frequently encountered cases of limit distributions.

Stable distributions are precisely those limit distributions that can occur in Generalized Central Limit theorems. Their study was begun by the mathematician P. Lévy in the 1920's. A recent book by Zolotarev and Uchaikin [10] contains over 200 pages of applications of stable distributions in probabilistic models, correlated systems and fractals, anomalous diffusion and chaos, physics, radiophysics, astrophysics, stochastic algorithms, financial applications, biology and geology. Stable distributions provide solutions to certain ordinary and fractional differential equations. The breadth of their applications suggests that they can be considered as a new class of special functions [9, 10, 13].

The Fourier transformed density distribution is usually called the characteristic function in mathematical statistics. The stable distributions are frequently given in terms of their characteristic functions. The reason for this is that the Fourier transform of a convolution is a product of the Fourier-transforms,

$$\tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q) \tag{11}$$

and limit distributions appear when the convolution of one additional elementary process does not change the shape of the limit distribution, but it results only in a modification of the parameters of the limit distribution. Hence, the stable distributions have simple characteristic functions. However, the explicit formulas describing the Lévy stable source density distributions are known only in some special cases. As of now, this is not an essential limitation as public domain numerical packages exist that can be utilized to calculate these source densities for any set of parameters [8].

Results of mathematical statistics yield a simple form for the characteristic function of univariate and symmetric stable distributions,

$$\tilde{f}(q) = \exp\left(iq\delta - |\gamma q|^{\alpha}\right) \tag{12}$$

where the support of the density function $f(x)$ is $(-\infty,\infty)$. Deep mathematical results imply that the index of stability, α, satisfies the inequality $0 < \alpha \leq 2$. This parameter determines, for large modulus of the coordinates, the Lévy distributions. Lévy laws with index of stability α tend to power-laws and the exponent of the decay of these distributions is given by $1 + \alpha$, $f(x) \to |x|^{-1-\alpha}$ for $|x| \to \infty$.

Although the proof that the shape given in eq. (12) is unique and that $0 < \alpha \leq 2$ is rather complicated, it is easy to show that these Lévy distributions are indeed stable under convolutions,

$$\begin{cases} \tilde{f}(q) = \exp\left(iq\delta - |\gamma q|^{\alpha}\right) \\ \prod_{i=1}^{n} \tilde{f}_i(q) = \exp\left(iq\delta_i - |\gamma_i q|^{\alpha}\right) \end{cases} \tag{13}$$

(14) $$\gamma^{\alpha} = \sum_{i=1}^{n} \gamma_i^{\alpha}, \quad \delta = \sum_{i=1}^{n} \delta_i$$

so after appropriate shifting and rescaling the stable distributions remain invariant. Observe that eq. (14) generalizes the well known quadratic addition rule of variances of convoluted Gaussian distributions to the case of stable distributions.

In the following, let us adopt the notation of Nolan [9]. Our choice corresponds to the $S(\alpha,\beta = 0, \gamma = R/2^{1/\alpha}, \delta = x_0;1)$ convention. In order to simplify the results, and to present results that are similar to the ones used in data fitting in high energy and nuclear physics, we have re-scaled the scale parameter γ of the Lévy distributions and introduced a physical notation as follows:

(15) $$R = 2^{1/\alpha}\gamma \quad x_0 = \delta \quad (\text{if } \beta = 0).$$

In the chosen $S(\alpha,\beta,\gamma,\delta;1)$ convention for symmetric stable distributions, the parameter δ coincides with x_0, the location parameter of the distribution, characterising the position of particle production. (This parameter is irrelevant in Bose-Einstein correlation studies, we shall see that it cancels from both the two- and the three-particle correlation functions.) In this notation, the one dimensional symmetric Lévy-stable distribution yields the following, simple form of the two-particle BEC:

(16) $$C(q) = 1 + \exp(-|qR|^{\alpha}).$$

This form that has an additonal parameter, the index of stability α, as compared to the usual Gaussian (or exponential) distribution, where the value of α is fixed to 2 (or 1). This result can be generalized straightforwardly to multidimensional expanding, core-halo type of systems. Two examples are given here, for more throughout discussion see Ref. [3].

Example a). For collisions with non-relativistic energy, and a small duration of particle emission, symmetric Lévy distributions yield the following Bose-Einstein correlation function [3]:

(17) $$C_2(k_1,k_2) = 1 + \lambda \exp\left[-\left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{\alpha/2}\right].$$

As usual, core-halo corrections [6, 10] introduce the intercept parameter λ. The three-dimensional expansion of the core results in a multivariate decomposition of q. All the fit parameters may depend on the mean momentum, $(\lambda, R_{ij}^2, \alpha) = (\lambda(\mathbf{K}), R_{ij}^2(\mathbf{K}), \alpha(\mathbf{K}))$, where $\mathbf{K} = 0.5(\mathbf{k}_1 + \mathbf{k}_2)$.

Example b). For very high energy collisions, the particle emission process becomes a highly relativistic phenomenon. In this case, the invariance of the emission function can be reflected if the longitudinally boost-invariant proper-time variable $\tau = \sqrt{t^2 - r_z^2}$ is utilized, and the space-time rapidity $\eta = 0.5\log[(t + r_z)/(t - r_z)]$ is also introduced as a hyperbolic, boost-additive coordinate.

In a factorized form, the Buda-Lund (BL) parameterization assumes the following structure for the emission function [4]:

(18) $$S(x,k) = H_*(\tau) G_*(\eta) I_*(r_x,r_y)$$

where the subscript * denotes an implicit momentum dependence. The effective proper-time and space-time rapidity distributions $H_*(\tau)$ and $G_*(\eta)$ are assumed to have a uni-variate Lévy distributions with indexes of stability $\alpha_=$ and $\alpha_\parallel$, while $I_*(r_x,r_y)$ may have a bivariate Lévy distribution with index $\alpha_\perp$. We use the symbolic notation [2] of the invariant temporal and the parallel relative momentum differences, $Q_=$ and $Q_\parallel$, being conjugated variables to the space-time variables (τ,η). In these variables, the correlation function is

(19) $$C(k_1,k_2) = 1 + \lambda \exp\left(-|R_= Q_=|^{\alpha_=} - |R_\parallel Q_\parallel|^{\alpha_\parallel} - |R_\perp Q_\perp|^{\alpha_\perp}\right)$$

(20) $$Q_= = m_{t,1}\cosh(y_1 - \bar{\eta}) - m_{t,2}\cosh(y_2 - \bar{\eta})$$

(21) $$Q_\perp = m_{t,1}\sinh(y_1 - \bar{\eta}) - m_{t,2}\sinh(y_2 - \bar{\eta})$$

(22) $$Q_\perp = \sqrt{Q_x^2 + Q_y^2}.$$

In these equations, $m_{t,i} = \sqrt{m^2 + \mathbf{k}_i^2}$ is the transverse mass, $y_i = 0.5\ln[(E_i + k_{z,i})/(E_i - k_{z,i})]$ is the rapidity of particle i and the fit parameter $\bar{\eta}$ stands for the space-time rapidity of the point of maximum emissivity for particles with a given fixed four-momentum k_i [4, 10]. The three different indexes of stability satisfy the usual unequality $0 < \alpha_i \le 2$ for all $i = (=,\parallel,\perp)$. All the five fitted scale parameters, λ, $R_=$, $R_\parallel$, $R_\perp$ and $\bar{\eta}$, as well as the three Lévy indexes $\alpha_=$, $\alpha_\parallel$ and $\alpha_\perp$ may depend on the value of the mean momentum $\mathbf{K}$.

Finally, let us consider the case of three-particle Bose-Einstein correlations. If the particle emission is completely chaotic and the plane-wave approximation can be warranted, this reads as

(23) $$C_3(1,2,3) = 1 + |\tilde{f}(1,2)|^2 + |\tilde{f}(2,3)|^2 + |\tilde{f}(3,1)|^2 + 2\Re \tilde{f}(1,2)\tilde{f}(2,3)\tilde{f}(3,1).$$

where the symbolic notation $\tilde{f}(i,j) \equiv \tilde{f}(k_i - k_j) \equiv \tilde{f}(q_{ij})$ has been introduced to simplify the equation. The three-particle cumulant correlation function corresponds to the last term,

(24) $$\kappa_3(1,2,3) = 2\Re \tilde{f}(1,2)\tilde{f}(2,3)\tilde{f}(3,1)$$

where the two-particle cumulant correlation function is defined as

(25) $$\kappa_3(1,2) = |\tilde{f}(1,2)|^2.$$

Hence, the normalized and symmetrized ratio

(26) $$\varpi(1,2,3) = \frac{\kappa_3(1,2,3)}{2\sqrt{\kappa_2(1,2)\kappa_2(2,3)\kappa_2(3,1)}}$$

turns out to be a simple function of β, the asymmetry parameter:

(27) $$\varpi(1,2,3) = \cos\varphi$$

(28) $$\varphi = \left\{ \frac{\beta}{2} R^{\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \left[\sum_{(i,j)} \left|q_{ij}\right|^{\alpha} \operatorname{sign}\left(q_{ij}\right) \right] \right\} \text{ for } \alpha \neq 1$$

(for the special case of $\alpha = 1$, see again Ref. [9].) In the above equation, the summation is taken over the cyclic permutations, $(i,j) = (1,2)$, $(2,3)$, or $(3,1)$. Note that the displacement parameter $\delta = x_0$ cancels from this result.

In eq. (28) all parameters are determined from the two-particle correlation function with the exception of β. This parameter is limited to the range of $-1 \leq \beta \leq 1$, it is called the asymmetry parameter of Lévy distributions, and can thus be determined from the relative momentum dependence of the normalized three-particle cumulant correlation function ϖ. Note that symmetric stable distributions correspond to the case of $\beta = 0$, hence in that case $\varphi = 0$ and $\varpi(1,2,3) = 1$ even at large relative momenta. Thus, all the essential parameters, α, β, and $\gamma = R/2^{1/\alpha}$ of stable source densities can be reconstructed from two and three particle correlation data. Let us emphasize that the result of eq. (28) is valid only within the plane-wave approximation and neglecting possible partial coherence in the source [2, 5].

In summary, we have determined the generic structure of Bose-Einstein correlations for the case when the coordinates of particle emission are obtained as convolution of many elementary subprocesses. Such processes can be attributed to parton-parton collisions, jet fragmentation, hadronization, rescattering and decay of hadronic resonances.

Our choice of multiple convolution of various elementary probability laws was motivated by a recent paper by A. Bialas [1], which considered Bose-Einstein correlations for the case, when the radius of a Gaussian source fluctuates from event to event. Numerically, similar results were obtained by Utyuzh, Wilk and Włodarczyk in Ref. [11] when considering Bose-Einstein correlations for sources with a fractal, power-law structure in space-time.

We find that the general shape of the Bose-Einstein correlation functions is a stretched exponential form. This is an exact analytic result, valid for all values of the relative momentum q, if the particle production is described by a stable law. Thus a new, experimentally measurable parameter is introduced to HBT or Bose-Einstein correlation studies, the Lévy index of stability, α. The corresponding Lévy stable distributions decay in the coordinate space as $|x|^{-1-\alpha}$ or large values of $|x|$ see Ref. [3] for greater details.

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